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Multiple Intersection Local Time of Planar Brownian Motion as a Particular Hida Distribution

Peter Imkeller

Laboratoire de Mathématiques, Université de Franche-Comté, 16, route de Gray, 25030 Besançon, France

and

Jia-An Yan

Institute of Applied Mathematics, Academia Sinica, 100080 Beijing, People's Republic of China

Subspaces of the space of Hida distributions are quantified in which the multiple intersection local time of planar Brownian motion, renormalized by taking off the term of order zero in its chaos decomposition, lives. This is done by means of a new formula expressing the expectation of higher products of multiple Wiener–Ito-integrals by certain functions of the scalar products of their kernels.

INTRODUCTION

For the construction of certain random fields in Euclidean quantum field theory, self intersection local times of multidimensional Brownian motion have proved to be useful (see Wolpert [18], Szymanzik [14]). This discovery created a considerable interest, especially among probalists, in self intersections of Brownian motion and their local times over a period of more than 20 years (see for example Imkeller *et al.* [6] for references). In recent years a new method of investigation of objects of this kind has joined the various more classical methods employed in their study: chaos or Fock space decompositions of Donsker's delta function, formulated in terms of Malliavin's calculus, or, even more appropriately, white noise calculus (see Hilda *et al.* [5]), have given rise to a powerful approach of asymptotic and renormalization properties of multiple intersection local times.

In Imkeller *et al.* [6] and Nualart and Vives [10], the double intersection local time of planar Brownian motion, renormalized by subtracting the term

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of order 0 in its chaos decomposition, was seen to be a Watanabe functional, more precisely an element of the quadratic Sobolev spaces of order α with respect to the Dirichlet structure defined by the Ornstein–Uhlenbeck operator for $\alpha < 1$. Recently Albeverio *et al.* [1] proved that it is not H-differentiable; i.e., it does not belong to the Sobolev space of order 1. Watanabe [15, 16], and He *et al.* [4] showed that the renormalized double intersection local time of d-dimensional Brownian motion is a Hida distribution, and gave a rule depending on d and stating which terms in the chaos decomposition have to be taken off to achieve a proper renormalization. White noise calculus was applied in Shieh [13] to show that k-fold intersection local times of planar Brownian motion for $k \geqslant 2$, renormalized by taking off the term of order 0 in the Fock space representation, is a Hida distribution. Moreover, the renormalized multiple intersection local time is represented in a formula of the Tanaka–Rosen–Yor type by stochastic integrals. But in this paper multiple intersections are allowed to occur only in a restricted sense: the k different times at which a k-fold intersection can happen have to be separated by strictly positive bounds.

In this paper we show that this latter restriction not only is entirely unnecessary, so that all possible k-fold intersections can be taken into account. We can also determine more precisely a subspace of the space of Hida distributions in which for $k \geqslant 2$ the k-fold intersection local time of planar Brownian motion lives: the dual of the space $\mathcal{G}_c$ of test functionals ϕ with chaos decompositions $(f_n)_{n \in \mathbf{N}}$ such that

$$\sum_{n=0}^{\infty} c^{2n} n! \, |f_n|^2 < \infty,$$

for any $c > k-1$ (Theorem 2). This is deduced by exploiting a new formula in which the expectation of multiple products of Wiener–Ito integrals $I_{n_j}(f_j^{\otimes n_j})$, $1 \leqslant j \leqslant l$, is expressed by simple functions of the scalar products $\langle f_i, f_j \rangle$, $i \neq j$ (Theorem 1).

In particular, the generality of this formula suggests that the results of this paper are not only interesting in their own right. Similar studies promise to be possible not only for dimensions larger than 2, but also for powers of double intersection local time instead of multiple intersection local time. Now these are objects which are likely to play an important role in an approach of Westwater's model of polymer measures (see Westwater [17]). This way one could possibly end up being able to explain more concisely the construction of the three dimensional polymer measure in Bolthausen [2], or at least to shed some new light on it.

Also, smoothness properties of intersection local times and similar functionals as the ones proved in this paper might open an access to capacitary results about local fine structure properties of Brownian motion.

1. PRELIMINARIES AND NOTATIONS

For $d \in \mathbf{N}$, we consider the Gel'fand triples

$$\mathscr{S}(\mathbf{R}) \subset L^2(\mathbf{R}) \subset \mathscr{S}^*(\mathbf{R}),$$

where $\mathscr{S}^*(\mathbf{R})$ is the dual of the space $\mathscr{S}(\mathbf{R})$ of rapidly decreasing functions, and

$$\mathscr{S}^d \subset L^d \subset (\mathscr{S}^d)^*,$$

where $L^d = L^2(\mathbf{R}; \mathbf{R}^d)$, a space we can identify with $L^2(\mathbf{R})^d$, and $\mathscr{S}^d$, $(\mathscr{S}^d)^*$ correspondingly (for more details see Hida *et al.* [5, Appendix 5]). λ denotes Lebesgue measure on the Borel sets of $\mathbf{R}$. Our basic probability space will be $((\mathscr{S}^d)^*, \mathscr{B}^d, P)$ with the Borel sets $\mathscr{B}^d$ of $(\mathscr{S}^d)^*$, and the d-fold product P of one-dimensional white noise measures on $\mathscr{S}^*(\mathbf{R})$. The dual pairing $\langle .,. \rangle$ between $\mathscr{S}^d$ and $(\mathscr{S}^d)^*$ is given by the sum of coordinatewise pairing of $\mathscr{S}(\mathbf{R})$ and $\mathscr{S}^*(\mathbf{R})$ which is denoted by the same symbol. The same convention will be in force for the Euclidean norm $|\cdot|$ generated by the pairings restricted to L^d resp. $L^2(\mathbf{R})$. For $h = (h_1, ..., h_d) \in \mathscr{S}^d$ we let

$$I_1(h)(\omega) = \langle h, \omega \rangle, \qquad \omega = (\omega_1, ... \omega_d) \in (\mathscr{S}^d)^*.$$

The mapping

$$\mathscr{S}^d \ni h \mapsto I_1(h) \in L^2((\mathscr{S}^d)^*, \mathscr{B}^d, P)$$

has a unique linear isometric extension to L^d. The image of h under this extension is still denoted by $I_1(h)$. In particular in this model the jth component W^j of the d-dimensional Wiener process $W = (W^1, ..., W^d)$ indexed by $\mathbf{R}$ can be obtained in the usual way as

$$W^j_t = I_1(h_t), \qquad \text{where} \quad h_t = (0, ..., 1_{[0 \wedge t, 0 \vee t]}, 0, ..., 0),$$

and the indicator function appears in the jth component of the vector. Actually, in the main sections, we shall consider the restriction of W to $[0, 1]$ as parameter space. The probability density of W_t will be denoted by

$$p^d_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{1}{2t}|x|^2\right), \qquad x \in \mathbf{R}^d, \quad t \in \mathbf{R}_+.$$

It is immediate that for $g \in L^d$ the usual Ito integral of deterministic functions coincides with $I_1(g)$. If I^j_1 denotes the Ito integral corresponding to W^j, $1 \leqslant j \leqslant d$, we have the equation

$$I_1(g) = \sum_{j=1}^{d} \int g_j \, dW^j = \int g \, dW = \sum_{j=1}^{d} I^j_1(g_j).$$

Now let

$$h_0(x) = 1, \qquad x \in \mathbf{R},$$

and for $n \in \mathbf{N}$

$$h_n(x) = \frac{\partial^n}{\partial u^n} \left[\exp\left(ux - \frac{1}{2} u^2 \right) \right] \Big|_{u=0}, \qquad x \in \mathbf{R},$$

the Hermite polynomial of degree n. For $f \in L^d$, $n \in \mathbf{N}$, we denote by $f^{\otimes n}$ the n-fold tensor product of f with itself, and set $I_0(c) = c$ for $c \in \mathbf{R}$, and

$$I_n(f^{\otimes n}) = |f|^n \, h_n \left(\frac{1}{|f|} I_1(f) \right).$$

Let $(e_i)_{i \in \mathbf{N}}$ be an ONB (orthonormal basis) of $L^2(\mathbf{R})$, and

$$e_i^j = (0, ..., 0, e_i, 0, ..., 0)$$

the vector with e_i as jth component, $1 \leqslant j \leqslant d$. Then $\{e_i^j : i \in \mathbf{N}, 1 \leqslant j \leqslant d\}$ is an ONB of L^d. For $i \in \mathbf{N}$, $1 \leqslant j \leqslant d$, we set

$$I_n((e_i^j)^{\otimes n}) = I_n^j(e_i^{\otimes n}), \qquad n \in \mathbf{N}.$$

If $\Lambda = \{a = (a_i)_{i \in \mathbf{N}} : a_i \in \mathbf{N}_0$ for $i \in \mathbf{N}$, $a_i = 0$ for almost all $i \in \mathbf{N}\}$, and for $a = (a_1, ..., a_N, 0, ...) \in \Lambda$

$$a! = \prod_{i=1}^{N} a_i!, \qquad |a| = \sum_{i=1}^{N} a_i,$$

then it is obvious that the family of symmetrizations of

$$\{e_1^{\otimes a_1} \otimes \cdots \otimes e_N^{\otimes a_N} : a = (a_1, ..., a_N, 0, ...) \in \Lambda\}$$

yields a COS (complete orthogonal system) of $L^2(\mathbf{R})^{\hat{\otimes} n}$ ($\hat{\otimes}_n$ stands for the symmetrized tensor product), where $n = |a|$. Here and henceforth we put $e_j^{\otimes 0} = 1$. For $n = |a|$, $1 \leqslant j \leqslant d$, we may define

$$I_n^j(e_1^{\otimes a_1} \otimes \cdots \otimes e_N^{\otimes \alpha_N}) = \prod_{i=1}^{N} I_{a_i}^j(e_i^{\otimes a_i}),$$

and extend this definition via linearity and isometry to all of $L^2(\mathbf{R})^{\hat{\otimes} n}$. This way one obtains the well known multiple Wiener–Ito integrals I_n^j, $n \in \mathbf{N}$, with respect to the 1-dimensional Wiener process W^j (see for example Nualart [8, pp. 12–17]).

To get corresponding objects w.r.t the d-dimensional Wiener process W, we just have to use the whole ONB of L^d instead. For $a^1, ..., a^d \in \Lambda$, $m \in \sum_{j=1}^d |a^j|$, $a_i^j = 0$ for $i > N$, $1 \leqslant j \leqslant d$, we let

$$I_m\left(\bigotimes_{j=1}^d \bigotimes_{i=1}^N (e_i^j)^{\otimes a_i^j}\right) = \prod_{j=1}^d \prod_{i=1}^N I_{a_i^j}((e_i^j)^{\otimes a_i^j}).$$

We use the fact that the family of symmetrizations of

$$\left\{\bigotimes_{j=1}^d \bigotimes_{i=1}^N (e_i^j)^{\otimes a_i^j} : a^j = (a_1^j, ..., a_N^j, 0, ...) \in \Lambda, N \in \mathbf{N}, 1 \leqslant j \leqslant d\right\}$$

is a COS of $(L^d)^{\hat{\otimes} m}$, linearity, and isometry to extend I_m to all elements of $(L^d)^{\hat{\otimes} m}$. The image of $(L^d)^{\hat{\otimes} m}$ is a closed subspace of $L^2((\mathscr{S}^d)^*, \mathscr{B}^d, P)$ we call mth *Wiener* chaos. Note that

$$I_m\left(\bigotimes_{j=1}^d \bigotimes_{i=1}^N (e_i^j)^{\otimes a_i^j}\right) = \prod_{j=1}^d I_{|a^j|}^j\left(\bigotimes_{i=1}^N e_i^{\otimes a_i^j}\right),$$

which clearly implies for $f \in L^d$, $n = (n_1, ..., n_d) \in \mathbf{N}_0^d$ that

$$I_{|n|}\left(\bigotimes_{j=1}^d ({}^j f)^{\otimes n_j}\right) = \prod_{j=1}^d I_{n_j}^j((f_j)^{\otimes n_j}), \tag{1}$$

where ${}^j f = (0, ..., 0, f_j, 0, ..., 0)$, $1 \leqslant j \leqslant d$, $|n| = \sum_{j=1}^d n_j$. We note moreover that

$$n! = \prod_{j=1}^d n_j!, \qquad h_n(x) = \prod_{j=1}^d h_{n_j}(x_j), \qquad x^n = \prod_{j=1}^d x_j^{n_j}, \quad x \in \mathbf{R}^d.$$

To abbreviate, for $f, g \in L^d$, we write

$$f^{\otimes n} = \bigotimes_{j=1}^d ({}^j f)^{\otimes n_j}$$

and

$$\langle f, g \rangle^n = \langle f^{\otimes n}, g^{\otimes n} \rangle = \prod_{j=1}^d \langle f_j, g_j \rangle^{n_j}.$$

It is well known that for each $\phi \in L^2((\mathscr{S}^d)^*, \mathscr{B}^d, P)$ possesses a chaos decomposition or Fock space representation

$$\phi = \sum_{n=0}^\infty I_n(f_n), \qquad f_n \in (L^d)^{\hat{\otimes} n}, \quad n \in \mathbf{N}_0$$

(here $(L^d)^{\hat{\otimes} 0} = \mathbf{R}^d$), which we write symbolically $\phi \sim (f_n)$. By orthogonality and isometry

$$\|\phi\|_2^2 = \sum_{n=0}^{\infty} \|I_n(f_n)\|_2^2 = \sum_{n=0}^{\infty} n!\, |f_n|^2,$$

where $\|\cdot\|_2$ stands for the L^2-norm in $L^2(\mathscr{S}^d)^*, \mathscr{B}^d, P)$. For $c > 1$ let

$$\mathscr{G}_c = \left\{\phi\colon \phi \sim (f_n) \in L^2((\mathscr{S}^d)^*, \mathscr{B}^d, P)\colon \sum_{n=0}^{\infty} c^{2n} n!\, |f_n|^2 < \infty\right\}.$$

If we take

$$\mathscr{G} = \bigcap_{c \geqslant 1} \mathscr{G}_c,$$

we obtain a test function space lying between the Schwartz and the Hida functions in the notation of Hida *et al.* [5]:

$$(\mathscr{S}) \subset \mathscr{G} \subset \mathscr{D}.$$

See Pothoff and Timple [11] and Yan [19]. $\mathscr{G}_c$ corresponds to $(E)_{0,\,c}$ of Yan [19] and $\mathscr{G}_{\ln c}$ of Pothoff and Timpel [11]. Any Hida distribution $\phi \in (\mathscr{S})^*$ can be identified with an element of Fock space $\phi \sim (g_n)$, $g_n \in ((\mathscr{S}^d)^*)^{\hat{\otimes} n}$, $n \in \mathbf{N}_0$. It is not hard to see that the dual of $\mathscr{G}_c$ can be identified with

$$\mathscr{G}_c^* = \left\{\phi \in (\mathscr{S})^*\colon \phi \sim (g_n),\ \sum_{n=0}^{\infty} c^{-2n} n!\, |g_n|^2 < \infty\right\},$$

$c > 1$. Taking

$$\mathscr{G}^* = \bigcup_{c \geqslant 1} \mathscr{G}_c^*,$$

we obtain the chain of inclusions

$$(\mathscr{S}) \subset \mathscr{G} \subset \mathscr{D} \subset L^2((\mathscr{S}^d)^*, \mathscr{B}^d, P) \subset \mathscr{D}^* \subset \mathscr{G}^* \subset (\mathscr{S})^*$$

(see Pothoff and Timple [11] and Yan [19]).

Dirac's delta distribution at $x \in \mathbf{R}^d$ is denoted by δ_x. $(\mathbf{N}_0^d)^*$ stands for $\{n \in \mathbf{N}_0^d\colon |n| \geqslant 1\}$.

2. A FORMULA FOR MOMENTS OF MULTIPLE WIENER–ITO INTEGRALS

We mention once more that from now on we restrict our parameter space to $[0, 1]$. Let $f^1, ..., f^k \in L^2([0, 1]^d$. The purpose of this short section is to give a compact formula for

$$E\left(\prod_{i=1}^{k} I_{|n_i|}((f^i)^{\otimes n_i})\right), \qquad n_i \in \mathbf{N}_0^d, \quad 1 \leqslant i \leqslant k.$$

To this end, we note first that according to the conventions above we have for $n \in \mathbf{N}_0^d$, $f = (f_1, ..., f_d) \in L^2([0, 1])^d$, $|f_j| = 1$, $1 \leqslant j \leqslant d$,

$$h_n(I_1^1(f_1), ..., I_1^d(f_d)) = I_{|n|}(f^{\otimes n}). \tag{2}$$

Let $\mathcal{M}_k$ be the set of all symmetric $k \times k$ matrices with entries in $\mathbf{N}_0$ and which vanish on the diagonal. For $\alpha \in \mathcal{M}_k$ let $\bar{\alpha}_i = \sum_{j=1}^{k} \alpha_{ij}$, $1 \leqslant i \leqslant k$, and $\bar{\alpha} = (\bar{\alpha}_1, ..., \bar{\alpha}_k)$.

Theorem 1. *Let $n = (n_1, ..., n_k) \in \mathbf{N}_0^{dk}$, $f^1, ..., f^k \in L^2([0, 1])^d$. Then we have*

$$E\left(\prod_{i=1}^{k} \frac{1}{n_i!} I_{n_i}((f^i)^{\otimes n_i}\right) = \sum_{\alpha \in \mathcal{M}_k^d,\, \bar{\alpha} = n} \prod_{p<q} \frac{1}{\alpha_{pq}!} \langle f^p, f^q \rangle^{\alpha_{pq}}.$$

Proof. By independence of the coordinates of the Wiener process, it is obviously enough to consider the case $d = 1$. Moreover, by scaling on both sides of the equation, we may suppose that $|f^j| = 1$ for $1 \leqslant j \leqslant d$. The vector

$$X = (I_1(f^1), ..., I_1(f^k))$$

is a Gaussian vector with covariance matrix given by

$$C = (\langle f^p, f^q \rangle)_{1 \leqslant p,\, q \leqslant k}.$$

Hence for any $u \in \mathbf{R}^k$

$$\begin{aligned} &E\left(\exp\left(\langle u, X \rangle - \frac{1}{2}|u|^2\right)\right) \\ &\quad = \frac{1}{(2\pi)^{k/2}} \int_{\mathbf{R}^k} \exp\left(-\frac{1}{2}|u|^2 - \frac{1}{2}|y|^2 + u^{\mathrm{T}} \sqrt{C}\, y\right) dy \\ &\quad = \exp\left(\frac{1}{2} u^{\mathrm{T}} [C - I_k] u\right), \end{aligned} \tag{3}$$

where I_k denotes the $k \times k$-identity matrix. Now we expand the right hand side of (3) into a power series. We have

$$\tfrac{1}{2} u^{\mathrm{T}} [C - I_k] u = \sum_{p<q} u_p c_{pq} u_q,$$

and hence

$$\begin{aligned}
\exp\left(\frac{1}{2} u^{\mathrm{T}} [C - I_k] u\right) &= \sum_{l=0}^{\infty} \frac{1}{l!} \left(\sum_{p<q} u_p c_{pq} u_q\right)^l \\
&= \sum_{\alpha \in \mathscr{M}_k} \prod_{p<q} \frac{1}{\alpha_{pq}!} u_p^{\alpha_{pq}} c_{pq}^{\alpha_{pq}} u_q^{\alpha_{pq}} \\
&= \sum_{\alpha \in \mathscr{M}_k} \prod_{p<q} \frac{1}{\alpha_{pq}!} c_{pq}^{\alpha_{pq}} \prod_{p=1}^{k} u_p^{\bar{\alpha}_p}.
\end{aligned}$$

A different expansion of (3) is obtained via the generating function of the Hermite polynomials. We have

$$\begin{aligned}
\exp\left(\langle u, X\rangle - \frac{1}{2} |u|^2\right) &= \sum_{n \in \mathbf{N}_0^k} \prod_{i=1}^{k} \frac{u_i^{n_i}}{n_i!} h_{n_i}(I_1(f^i)) \\
&= \sum_{n \in \mathbf{N}_0^k} \prod_{i=1}^{k} \frac{u_i^{n_i}}{n_i!} I_{n_i}((f^i)^{\otimes n_i}). \qquad (4)
\end{aligned}$$

It remains to take expectations in (4) and compare the resulting power series term by term with (3) to get the desired formula. ∎

3. MULTIPLE INTERSECTION LOCAL TIME OF PLANAR BROWNIAN MOTION

Let in this section $d = 2$. Our approach is based on the chaos decomposition of Donker's delta distribution. For $h \in L^2([0,1])^2$, its 2-dimensional version according to Hida *et al.* [5, p. 47], or Imkeller *et al.* [6] states

$$\delta_0(I_{(1,1)}(h)) = \sum_{n \in \mathbf{N}_0{}^2} \frac{1}{(2n)!} h_{2n}(0)\, I_{2|n|}\left(\left[\left(\frac{h_1}{|h_1|}, \frac{h_2}{|h_2|}\right)\right]^{\otimes 2n}\right) p^1_{|h_1|^2}(0)\, p^1_{|h_2|^2}(0).$$

Hence for $0 \leqslant t_1 < \cdots < t_{k+1} \leqslant 1$, with the abbreviations $J_i = [t_i, t_{i+1}]$, $f^i = (1_{J_i}/\sqrt{\lambda(J_i)}, 1_{J_i}/\sqrt{\lambda(J_i)})$, $1 \leqslant i \leqslant k$,

$$\prod_{i=1}^{k} \delta_0(W_{t_{i+1}} - W_{t_i}) = \sum_{n_1, \ldots, n_k \in \mathbf{N}_0^2} \prod_{i=1}^{k} \frac{1}{(2n_i)!} h_{2n_i}(0)\, I_{2|n_i|}((f^i)^{\otimes 2n_i})\, p^2_{\lambda(J_i)}(0). \qquad (5)$$

According to Hida *et al.* [5, p. 48], and independence of the increments of W, this series describes an element of the space of Hida distributions. Our main object of interest can now formally be given by the expression

$$\delta^k = \int_{\Delta_k} \prod_{i=1}^{k} \delta_0(W_{t_{i+1}} - W_{t_i})\, dt_1 \cdots dt_{k+1},$$

where $\Delta_k = \{(t_1, ..., t_{k+1}) \in [0, 1]: t_1 < \cdots < t_{k+1}\}$. We shall show that the following renormalized version of δ^k, again at first only formally given by

$$\bar{\delta}^k = \int_{\Delta_k} \prod_{i=1}^{k} \left[\delta_0(W_{t_{i+1}} - W_{t_i}) - \frac{1}{2\pi(t_{i+1} - t_i)} \right] dt_1 \cdots dt_{k+1},$$

exists as a Hida distribution, and determine more precisely its position in this distribution space. $\bar{\delta}^k$ will be called $k+1$-*fold intersection local time of* W. To achieve this aim, we shall expand $\bar{\delta}^k$ using (5) and the moment formula of theorem 1. This procedure will lead to the study of the following interaction integrals.

For $k, l \in \mathbf{N}$ let $\mathscr{S}_{k,l}$ be the set of k-tuples $(S_1, ..., S_k)$ of subsets of $\{k+1, ..., k+l\}$, covering this set and increasing in the sense $i \leqslant j$ for $i \in S_r$, $j \in S_{r+1}$, and such that $|S_r \cap S_{r+1}| = 1$, $1 \leqslant r \leqslant k-1$. The S_r will be, roughly speaking, intervals in $\{k+1, ..., k+l\}$, covering this set, increasing in r, and with an overlap of exactly one element. Dually, we let $\mathscr{T}_{k,l}$ be the set of l-tuples $(T_{k+1}, ..., T_{k+l})$ of covering subsets of $\{1, ..., k\}$, with analogous properties of increase and overlap. For $\alpha \in \mathscr{M}_{k+l}$ let

$$J(\alpha) = \int_{\Delta_{k,l}} \prod_{p<q} \left(\frac{\lambda(J_p \cap J_q)}{[\lambda(J_p)\, \lambda(J_q)]^{1/2}} \right)^{\alpha_{pq}} \prod_{i=1}^{k+l} \lambda(J_i)^{-1}\, dt_1 \cdots ds_{k+l+1},$$

where we put

$$J_i = [t_i, t_{i+1}] \text{ resp. } [s_i, s_{i+1}],$$

$$\text{for} \quad 1 \leqslant i \leqslant k \text{ resp. } k+1 \leqslant i \leqslant k+l,$$

and

$$\Delta_{k,l} = \{(t_1, ..., t_{k+1}, s_{k+1}, ..., s_{k+l+1})$$

$$\in [0, 1]^{k+l+1}: t_1 < \cdots < t_{k+1}, s_{k+1} < \cdots < s_{k+l+1}\}.$$

Moreover, for $S=(S_1, ..., S_k) \in \mathscr{S}_{k,l}$ let

$$J(\alpha, S)=\int_{\Delta_{k,l},\, \lambda(J_i \cap J_j)>0 \Leftrightarrow j \in S_i} \prod_{p<q} \left(\frac{\lambda(J_p \cap J_q)}{[\lambda(J_p)\, \lambda(J_q)]^{1/2}} \right)^{\alpha_{pq}} \times \prod_{i=1}^{k+l} \lambda(J_i)^{-1}\, dt_1 \cdots ds_{k+l+1},$$

and for $T=(T_{k+1}, ..., T_{k+l}) \in \mathscr{T}_{k,l}$ let $J(\alpha, T)$ be defined analogously. Then we have

$$J(\alpha)=\sum_{S \in \mathscr{S}_{k,l}} J(\alpha, S)=\sum_{T \in \mathscr{T}_{k,l}} J(\alpha, T). \tag{6}$$

We say that S and T are *associated* if we have

$$j \in S_i \qquad \text{iff} \quad i \in T_j, \quad 1 \leqslant i \leqslant k, \quad k+1 \leqslant j \leqslant k+l.$$

If S and T are associated, clearly

$$J(\alpha, S)=J(\alpha, T). \tag{7}$$

Finally, for $S \in \mathscr{S}_{k,l}$ let

$$\mathscr{M}_{k+l}(S)=\{a \in \mathscr{M}_{k+l} \colon \alpha_{pq}=0 \text{ unless } q \in S_p,\, 1 \leqslant p \leqslant k\}.$$

For $T \in \mathscr{T}_{k,l}$ we define $\mathscr{M}_{k+1}(T)$ analogously, and we remark that if S and T are associated, then

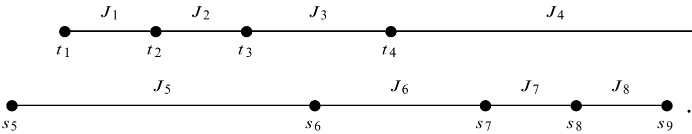
$$\mathscr{M}_{k+l}(S)=\mathscr{M}_{k+l}(T). \tag{8}$$

The formal definitions just given may be a little hard to interpret. We therefore discuss the main parts briefly. First of all, the sets $\mathscr{S}_{k,l}$ and $\mathscr{T}_{k,l}$ are defined to split the domain of integration of $J(\alpha)$ into handy parts. For $k \neq l$, we just need them for the sake of formal consistency in the proofs by induction in the following lemmas. Of course, our main interest is in $\mathscr{S}_{k,k}$ and $\mathscr{T}_{k,k}$. If a k-tuple $S=(S_1, ..., S_k)$ and its associate $T=(T_{k+1}, ..., T_{2k})$ are given, the domain of integration of $J(\alpha, S)$ is the set of all $2k+2$-tuples $(t_1, ..., t_{k+1}, s_{k+1}, ..., s_{2k+1})$ such that the t_i and the s_j are linearly ordered and such that $J_i=[t_i, t_{i+1}]$ resp. $[s_i, s_{i+1}]$ for $1 \leqslant i \leqslant k$ resp. $k+1 \leqslant i \leqslant 2k$ satisfy

$$\lambda(J_i \cap J_j) \neq 0 \qquad \text{iff} \quad j \in S_i \text{ (i.e. iff } i \in T_j),$$

$1 \leqslant i \leqslant k$, $k+1 \leqslant j \leqslant 2k$.

To fix the ideas, let us give an example in the case $k=4$. Let $S=(\{5\}, \{5\}, \{5, 6\}, \{6, 7, 8\})$, and thus $T=(\{1, 2, 3\}, \{3, 4\}, \{4\}, \{4\})$. Then we integrate over all 10-tuples $(t_1, ..., t_5, s_5, ..., s_9)$ which are in the following relative positions:



Note that s_5 may be on either side of t_1, and s_9 on either side of t_5. S and T may be viewed in diagrams which for our example take the form

$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & \\ 5 & \bullet & \bullet & \bullet & & T_5 \\ 6 & & & \bullet & \bullet & T_6 \\ 7 & & & & \bullet & T_7 \\ 8 & & & & \bullet & T_8 \\ & S_1 & S_2 & S_3 & S_4 & \end{array}.$$

LEMMA 1. *Let $S \in \mathcal{S}_{k,l}$, $\alpha \in \mathcal{M}_{k+l} \backslash \mathcal{M}_{k+l}(S)$. Then*

$$J(\alpha, S)=0.$$

Proof. This is an immediate consequence of the definition of $J(\alpha, S)$. The integrand becomes 0 if $\alpha_{pq} \neq 0$ for some $q \notin S_p$, $1 \leqslant p \leqslant k$. ∎

We shall now give bounds for $J(\alpha, S)$ with $\alpha \in \mathcal{M}_{k+l}(S)$.

LEMMA 2. *Let $S \in \mathcal{S}_{k,l}$, $\alpha \in \mathcal{M}_{k+l}(S)$ such that $\bar{\alpha}_i \in 2\mathbf{N}$ for $1 \leqslant i \leqslant k+l$. Then $\alpha_{ij} \in 2\mathbf{N}$ for $1 \leqslant i \leqslant k$, $j \in S_i$. If T is associated with S, then we have*

$$\begin{aligned} J(\alpha, S) &= J(\alpha, T) \\ &= \int_{\Delta_{k,l}} \prod_{i=1}^{k} \prod_{j \in S_i} \left(\frac{\lambda(J_i \cap J_j)}{[\lambda(J_i)\,\lambda(J_j)]^{1/2}} \right)^{\alpha_{ij}} \prod_{i=1}^{k+l} \lambda(J_i)^{-1}\, dt_1 \cdots ds_{k+l+1} \\ &= \int_{\Delta_{k,l}} \prod_{j=k+1}^{k+l} \prod_{i \in T_j} \left(\frac{\lambda(J_i \cap J_j)}{[\lambda(J_i)\,\lambda(J_j)]^{1/2}} \right)^{\alpha_{ij}} \prod_{i=1}^{k+l} \lambda(J_i)^{-1}\, dt_1 \cdots ds_{k+l+1}, \end{aligned}$$

and

$$J(\alpha, S) \leqslant 3 \prod_{i=1}^{k} \prod_{j \in S_i} \frac{2}{\alpha_{ij}}.$$

Proof. By the very definition of $\mathcal{S}_{k,l}$, $\mathcal{T}_{k,l}$, the second assertion is clear. For the first and third, we proceed by induction on $k+l$.

The first one is clear if $k=l=1$, for then $\bar{\alpha}_1=\bar{\alpha}_2=\alpha_{12}$. Assume it is true for $k+l\leqslant n$. To prove it for $k+l+1$, the symmetry of the roles of k and l allows us to suppose that $S=(S_1, ..., S_{k+1})\in\mathcal{S}_{k+1,l}$ and $T=(T_{k+1}, ..., T_{k+l+1})\in\mathcal{T}_{k+1,l}$ is associated with S. If $|S_{k+1}|=1$, the induction hypothesis can immediately by applied. If $|S_{k+1}|\geqslant 2$, say

$$S_{k+1}=\{r, r+1, ..., k+1+l\},$$

then by definition of $\mathcal{S}_{k+1,l}$ and $\mathcal{T}_{k+1,l}$, we have

$$T_{r+1}=\cdots=T_{k+1+l}=\{k+1\},$$

hence by hypothesis (remember that α is symmetric) $\alpha_{k+1j}\in 2\mathbf{N}$ for $r+1\leqslant j\leqslant k+1+l$, and, since again by hypothesis $\bar{\alpha}_{k+1}\in 2\mathbf{N}$, we obtain also $\alpha_{k+1r}\in 2\mathbf{N}$. It remains to apply the induction hypothesis to get the first statement.

Let us now derive the asserted inequality by induction as well. Suppose first that $k=l=1$. Then with $\beta=\alpha_{12}$, $S=(\{2\})$ we have

$$\begin{aligned}
J(\alpha, S)&=2\int_{\{t_1<s_2<t_2<s_3\}}\frac{(t_2-s_2)^{\beta}}{[(t_2-t_1)(s_3-s_2)]^{\beta/2+1}}\,dt_1\cdots ds_3\\
&\quad+2\int_{\{t_1<s_2<s_3<t_2\}}\frac{(s_3-s_2)^{\beta/2-1}}{(t_2-t_1)^{\beta/2+1}}\,dt_1\cdots ds_3\\
&=2[J_1+J_2].
\end{aligned}$$

To estimate J_1, we change coordinates by the rule

$$w=t_1,\qquad x=t_2-s_2,\qquad y=t_2-t_1,\qquad z=s_3-s_2.$$

Then

$$\begin{aligned}
J_1&\leqslant\int_{\{x\leqslant y\wedge z\}}\frac{x^{\beta}}{(yz)^{\beta/2+1}}\,dx\,dy\,dz\\
&\leqslant\frac{2}{\beta+1}\int_{\{y\leqslant z\}}\frac{y^{\beta/2}}{z^{\beta/2+1}}\,dy\,dz\\
&\leqslant\frac{4}{(\beta+1)(\beta+2)}.
\end{aligned}$$

For J_2, we use the coordinates

$$w=t_1, x=s_3-s_2, y=t_2-t_1, z=t_2-s_3.$$

Then

$$J_2 \leqslant \int_{\{x+z \leqslant y\}} \frac{x^{\beta/2-1}}{y^{\beta/2+1}}\, dx\, dy\, dz$$

$$\leqslant \int_{\{x \leqslant y\}} \frac{x^{\beta/2-1}}{y^{\beta/2}}\, dx\, dy$$

$$\leqslant \frac{2}{\beta}.$$

Summarizing, we obtain

$$J(\alpha, S) \leqslant \frac{6}{\beta}. \tag{9}$$

For the induction step, suppose the inequality is valid for some $k+l \geqslant 2$, and all $S \in \mathscr{S}_{k,\,l}$, $\alpha \in \mathscr{M}_{k+l}(S)$. To prove it for $k+l+1$, by symmetry we may assume as above that

$$S = (S_1, ..., S_{k+1}) \in \mathscr{S}_{k+1,\,l} \qquad \text{and} \qquad T = (T_{k+2}, ..., T_{k+l+1}) \in \mathscr{T}_{k+1,\,l}$$

are associated. Note first that

$$\lambda(J_{k+1} \cap J_{k+l+1}) > 0 < \lambda(J_{k+1} \cap J_{k+l}),$$

and

$$\lambda(J_{k+l+1} \cap J_{k+1}) > 0 < \lambda(J_{k+l+1} \cap J_k)$$

is impossible due to the linear ordering of the intervals $J_1, ..., J_{k+1}$ and $J_{k+2}, ..., J_{k+l+1}$. Consequently either $|S_{k+1}| = 1$ or $|T_{k+l+1}| = 1$. Symmetry allows us once more to suppose that $|S_{k+1}| = 1$. We now integrate first in t_{k+2}, leaving the remaining variables fixed. To abbreviate, we denote hereby

$$R_k = \prod_{i=1}^{k} \prod_{j \in S_i} \left(\frac{\lambda(J_i \cap J_j)}{[\lambda(J_i)\, \lambda(J_j)]^{1/2}} \right)^{\alpha_{ij}} \prod_{j \neq k} \lambda(J_j)^{-1},$$

and set

$$u = (t_1, ..., t_{k+1}, s_{k+1}, ..., s_{k+l+1}),$$

$$v_1 = t_{k+1}, \qquad v_2 = t_{k+2}, \qquad v_3 = s_{k+l}, \qquad v_4 = s_{k+l+1}, \qquad \beta = \alpha_{k+1\,k+l+1}.$$

Then, suppressing to mention the linear order in the components of u,

$$\begin{aligned} J(\alpha, S) &= \int_{\{v_4 < v_2\}} R_k \frac{(v_4 - v_1)^\beta}{[(v_2 - v_1)(v_4 - v_3)]^{\beta/2}} \frac{1}{v_2 - v_1} \, dv_2 \, du \\ &\quad + \int_{\{v_4 > v_2\}} R_k \frac{(v_2 - v_1)^{\beta/2 - 1}}{(v_4 - v_3)^{\beta/2}} \, dv_2 \, du \\ &= K_1 + K_2. \end{aligned}$$

To estimate K_1, observe that by the first part of the lemma $\beta \geqslant 2$. Hence we obtain upon integration in v_2

$$\begin{aligned} K_1 &\leqslant \frac{2}{\beta} \int R_k \frac{(v_4 - v_1)^{\beta/2}}{(v_4 - v_3)^{\beta/2}} \, du \\ &\leqslant \frac{2}{\beta} \int R_k \, du. \end{aligned}$$

For K_2, the argument is similar and the resulting inequality identical. Now let $S' = (S_1, ..., S_k)$, and α' the $(k+l) \times (k+l)$-matrix one gets from α by cutting off the $k+1$st line and column. Then $S' \in \mathscr{S}_{k,l}$, $\alpha' \in \mathscr{M}_{k+l}(S')$ and

$$\int R_k \, du = J(\alpha', S').$$

Therefore we may apply the induction hypothesis to complete the inductive argument. ∎

To estimate the coefficients in the expansion, we shall use the following lemma.

Lemma 3. *Let $k, l \in \mathbf{N}$, $S \in \mathscr{S}_{k,k}$, $\alpha \in \mathscr{M}^2_{2k}(S)$ such that*

$$\sum_{i=1}^{k} |\bar{\alpha}_i| = l \quad \left(= \sum_{i=k+1}^{2k} |\bar{\alpha}_i| \right).$$

Then

$$\prod_{p=1}^{k} \prod_{q \in S_p} \frac{1}{\alpha_{pq}!} \prod_{p=1}^{2k} h_{\bar{\alpha}_p}(0) \leqslant k^l.$$

Proof. We use the following simple and well known inequality for Hermite polynomials

$$|h_r(0)| \leqslant \sqrt{r!}, \qquad r \in \mathbf{N}. \tag{10}$$

Note that, if T is associated with S, we have

$$\prod_{p=1}^{k} \prod_{q \in S_p} \frac{1}{\alpha_{pq}!} = \prod_{q=k+1}^{2k} \prod_{p \in T_q} \frac{1}{\alpha_{pq}!}. \tag{11}$$

(10) and (11) imply the following chain of inequalities:

$$\begin{aligned}
&\prod_{p=1}^{k} \prod_{q \in S_p} \frac{1}{\alpha_{pq}!} \prod_{p=1}^{2k} h_{\bar{\alpha}_p}(0) \\
&\quad = \prod_{p=1}^{k} \prod_{q \in S_p} \frac{1}{\alpha_{pq}!} \cdot \prod_{q=k+1}^{2k} \prod_{p \in T_q} \frac{1}{\alpha_{pq}!} \\
&\quad \leqslant \prod_{p=1}^{k} \sqrt{\frac{\bar{\alpha}_p!}{\prod_{q \in S_p} (1/\alpha_{pq}!)}} \cdot \prod_{q=k+1}^{2k} \sqrt{\frac{\bar{\alpha}_q!}{\prod_{p \in T_q} (1/\alpha_{pq}!)}} \\
&\quad \leqslant \prod_{p=1}^{k} |S_p|^{|\bar{\alpha}_p|/2} \cdot \prod_{q=k+1}^{2k} |T_q|^{|\bar{\alpha}_q|/2} \\
&\quad \leqslant k^l.
\end{aligned} \tag{12}$$

For the third line in (12), we use the simple inequality

$$\binom{k}{k_1 \cdots k_m} \leqslant m^k,$$

valid for $k, m \in \mathbf{N}$, $k_1, ..., k_m \in \mathbf{N}_0$, which results from the multinomial theorem. Note that the quantity on the left hand side of (12) is nonnegative due to the fact

$$\sum_{i=1}^{k} |\bar{\alpha}_i| = \sum_{i=k+1}^{2k} |\bar{\alpha}_i|.$$

(12) is the asserted inequality. ∎

We are ready to classify $\bar{\delta}^k$ as a Hida distribution.

THEOREM 2. *For $l \in \mathbf{N}$, let*

$$\begin{aligned}
H_{2l} &= \sum_{\Gamma_k^l} \int_{\Delta_k} \prod_{i=1}^{k} \frac{1}{(2n_i)!} h_{2n_i}(0)\, I_{2|n_i|}((f^i)^{\otimes 2n_i})\, p^2_{\lambda(J_i)}(0)\, dt_1 \cdots ds_{2k+1} \\
&= \int_{\Delta_k} I_{2l}\left(\sum_{\Gamma_k^l} \prod_{i=1}^{k} \frac{1}{(2n_i)!} h_{2n_i}(0) \bigotimes_{i=1}^{k} (f^i)^{\otimes 2n_i} p^2_{\lambda(J_i)}(0) \right) dt_1 \cdots ds_{2k+1},
\end{aligned}$$

where

$$f^i = \left(\frac{1_{J_i}}{\sqrt{\lambda(J_i)}}, \frac{1_{J_i}}{\sqrt{\lambda(J_i)}} \right), J_i = [t_i, t_{i+1}], \qquad 1 \leqslant i \leqslant k,$$

and

$$\Gamma^l_k = \left\{ (n_1, ..., n_k) : n_i \in (\mathbf{N}_0^2)^*, 1 \leqslant i \leqslant k, \sum_{i=1}^k |n_i| = l \right\}.$$

Then

$$\bar{\delta}^k = \sum_{l=1}^\infty H_{2l} \in \mathscr{G}_c^* \qquad \textit{for any} \quad c > k.$$

For $l \in \mathbf{N}$, H_{2l} *is the projection of* $\bar{\delta}^k$ *on the* $2l$th *chaos.*

Proof. Since the intervals J_i, $1 \leqslant i \leqslant k$, are λ-almost everywhere disjoint, independence of increments of W allows us to write

$$\prod_{i=1}^k I_{2|n_i|}((f^i)^{\otimes 2n_i}) = I_{2l} \left(\bigotimes_{i=1}^k (f^i)^{\otimes 2n_i} \right).$$

Therefore it is clear that H_{2l} is located in the $2l$th chaos, and all we have to prove is

$$\sum_{l=1}^\infty c^{2l} \|H_{2l}\|_2^2 < \infty \qquad \text{for} \quad c < \frac{1}{k}. \tag{13}$$

For this purpose, we shall calculate $\|H_{2l}\|_2^2$ for $l \in \mathbf{N}$ using Theorem 1. Writing as above $J_i = [t_i, t_{i+1}]$ resp. $[s_i, s_{i+1}]$ for $1 \leqslant i \leqslant k$, resp. $k+1 \leqslant i \leqslant 2k$, and extending the definition of f^i for $1 \leqslant i \leqslant 2k$ in the obvious way, we have

$$\begin{aligned}
\|H_{2l}\|_2^2 = & \sum_{\substack{(n_1, ..., n_k) \in \Gamma^l_k, \\ (n_{k+1}, ..., n_{2k}) \in \Gamma^l_k}} \int_{\Delta_{k,k}} E\left(\prod_{i=1}^{2k} \frac{1}{(2n_i)!} h_{2n_i}(0)\, I_{2|n_i|}((f^i)^{\otimes 2n_i}) \right) \\
& \times p^2_{\lambda(J_i)}(0)\, dt_1 \cdots ds_{2k+1} \\
= & \sum_{\substack{(n_1, ..., n_k) \in \Gamma^l_k, \\ (n_{k+1}, ..., n_{2k}) \in \Gamma^l_k}} \int_{\Delta_{k,k}} \sum_{\alpha \in \mathscr{M}^2_{2k},\, \bar{\alpha} = n} \prod_{p<q} \frac{1}{\alpha_{pq}!} \langle f^p, f^q \rangle^{|\alpha_{pq}|} \\
& \times \prod_{i=1}^{2k} h_{2n_i}(0)\, p^2_{\lambda(J_i)}(0)\, dt_1 \cdots ds_{2k+1}
\end{aligned}$$

$$= (2\pi)^{-2k} \sum_{\alpha \in \mathcal{M}^2_{2k},\, \sum_{i=1}^k |\bar{\alpha}_i| = l} J(2\,|\alpha|) \prod_{p<q} \frac{1}{(2\alpha_{pq})!} \prod_{i=1}^{2k} h_{2\bar{\alpha}_i}(0)$$

$$= (2\pi)^{-2} \sum_{S \in \mathcal{S}_{k,k}} \sum_{\alpha \in \mathcal{M}^2_{2k}(S),\, \sum_{i=1}^k |\bar{\alpha}_i| = l} J(2\,|\alpha|) \prod_{p<q} \frac{1}{(2\alpha_{pq})!} \prod_{i=1}^{2k} h_{2\bar{\alpha}_i}(0)$$

$$\leqslant (2\pi)^{-2k} k^{2l} \sum_{S \in \mathcal{S}_{k,k}} \sum_{\alpha \in \mathcal{M}^2_{2k}(S),\, \sum_{i=1}^k |\bar{\alpha}_i| = l} \prod_{i=1}^{k} \prod_{j \in S_i} \frac{1}{\alpha_{ij}}.$$

The third line in (14) is due to Theorem 1, the fifth to Lemma 1, and the last to Lemmas 2, 3. Now, on the one hand,

$$\sum_{\beta_1, \beta_2 \in \mathbf{N}_0,\, \beta_1 + \beta_2 = r} \frac{1}{r} \leqslant 2,$$

for $r \in \mathbf{N}$, and, on the other hand,

$$\sum_{\beta_1, \ldots, \beta_p \in \mathbf{N},\, \sum_{i=1}^p \beta_i = l} 1 \leqslant l^p,$$

for $p, l \in \mathbf{N}$. Consequently for $S \in \mathcal{S}_{k,k}$ fixed

$$\sum_{S \in \mathcal{S}_{2k}} \sum_{\alpha \in \mathcal{M}^2_{2k}(S),\, \sum_{i=1}^k |\bar{\alpha}_i| = l} \prod_{i=1}^{k} \prod_{j \in S_i} \frac{1}{\alpha_{ij}} \leqslant 2^{2k} \cdot l^k,$$

and hence, with the constant $c_k = |\mathcal{S}_{k,k}|$, (14) yields the final inequality

$$\|H_{2l}\|_2^2 \leqslant 3\pi^{-2k} c_k k^{2l} l^k. \tag{15}$$

(15) evidently implies (13) and the proof is completed. ∎

Remark. In Shieh [13] multiple intersection local times of planar Brownian motion "off the diagonal" are considered; i.e., for $a_i, b_i \in [0, 1]$, $1 \leqslant i \leqslant k$, such that $a_1 < b_1 < \cdots < a_k < b_k$, fixed, the functional

$$\tilde{\delta}^k = \int_{\{a_i < t_i < t_{i+1} < b_i,\, 1 \leqslant i \leqslant k\}} \prod_{i=1}^{k} \delta_0(W_{t_{i+1}} - W_{t_i})\, dt_1 \cdots dt_{k+1}$$

is investigated, and proved to be a Hida distribution. Our Theorem 2 not only settles the question raised in the paper as to what happens if intersections "near the diagonal" are admitted, but also improves the "off-diagonal" result by specifying the subspaces $\mathcal{G}_c^*$ discussed above.

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